This lecture will cover parent functions, transformations, and order of transformations.
Parent functions are basic functions that can be used to create or build more complex functions. Six particular parent functions will be considered in this lecture, with additional parent functions saved for future lectures.

Three of the six parent functions are:
The identity function, also known as the linear parent function, the square function, also referred to as the quadratic parent function, and the cubic function.
Parent Functions

- *Parent functions* are simple functions that act as bases for other, complex functions.
  - Square root function
    \[ f(x) = \sqrt{x} \]
  - Cube root function
    \[ f(x) = \sqrt[3]{x} \]
  - Absolute value function
    \[ f(x) = |x| \]

The other three parent functions are:
The square root function, the cube root function and the absolute value function.
To sketch the graph of the identity function and the other parent functions, points on the graph using a t-chart need to be determined. These points can then be used to sketch the graph.

For the identity function, begin with the $x$-values -2, -1, 0, 1, 2. Notice that the function rule states that the function value, or $y$-value is the same as the $x$-value.

Each row in the t-chart represents a point on the graph. So after plotting the points, the graph should clearly a straight line, with the domain of the function being all real numbers and the range also being all real numbers.
To sketch the graph of this function, begin by considering the points on the graph corresponding to the $x$-values $-2, -1, 0, 1, 2$.

Remember that $(-2)^2 = -2 \times -2 = 4$, not $-4$.

Now plot the points and fill in the shape of the graph to see that the graph has a U-shape form.

The domain of this function is also all real numbers. However, note that there are no negative $y$-values in the graph of the square function. So the range consists of all nonnegative real numbers, represented in interval notation by $[0, \infty)$. 

\[ f(x) = x^2 \]

\begin{tabular}{c|c}
$x$ & $f(x)$ \\
\hline
\end{tabular}

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{square_function_graph}
\caption{Square Function Graph}
\end{figure}
The same strategy can be used to draw the graph of the cubic function: using the t-chart for this function. Remember that each x-value is cubed. So \((-2)^3 = -2 \times -2 \times -2 = -8\).

Notice the larger y-values. Unlike the square function, the cubic function has negative y-values. So the domain and range for the cubic function are both all real numbers.
The square root parent function differs from the first three parent functions in that only specific $x$-values can be used to find points for the graph: $x = -1$ and $-2$ cannot be used since the function values at these $x$-values are not real numbers. So only nonnegative values can be used. Remember that the graph of this function does not extend to the left beyond $x = 0$. So the domain is the set of all nonnegative values. The range is the same as the domain.
For the cube root function, the choice of x-values are not as restricted as the square root function. However, it is nice to choose x-values that have y-values that are integers.

Remember from the cubic function that \(2^3 = 8\). So \(x = 8\) can be used. As a matter of fact, all of the y-values from the cubic function can be used as the x-values for the cube root function.

The shape of the graph is similar to the form of the cubic graph. So clearly, from the graph, the domain and range of the cube root function are both represented by the set of all real numbers.
This parent function is easier to graph than either the square root or cube root functions.

Using the same set of $x$-values as with the graphs of the first three functions, be sure to make the distinction between this function’s graph and the graph of the square function. This function has a v-shape.

The absolute value function shares the same domain and range as the square function.
The first two type of transformations made to functions are vertical shifts.

If we take the function \( f(x) \) and add a real number \( k \) to the function value, then the graph of the function is shifted \( k \) units upward.

Similarly, if we subtract \( k \) from the function value, then the graph of the function is shifted \( k \) units downward.
Example: Graph \( g(x) = x^2 - 3 \).

Example:
Begin by identifying the parent function that has been transformed. Notice the square exponent. This is the indicator that the parent function is the square function, \( f(x) = x^2 \). So \( g(x) \) is of the form \( f(x) - k \). Hence, the graph of \( g(x) \) can be found by shifting the graph of the square function down 3 units.
Example: Graph $h(x) = \sqrt{x} + 1$.

Example:
Here the square root function is the parent function with the transformation being a shift upward by 1 unit.
To graph, start the graphing process by sketching the parent function.
Now, shift the points of this original graph upward one unit to sketch the graph of $h(x)$.
Notice that the domain remains the same, but the range has changed.
The graph of a function may also be shifted horizontally. Horizontal shifts behave somewhat differently than vertical shifts.

If $x$ is replaced with $x+h$ in the function $f(x)$, then the graph of $f(x)$ is shifted to the **LEFT** $h$ units.

If $x$ is replaced with $x-h$ in the function $f(x)$, then the graph of $f(x)$ is shifted to the **RIGHT** $h$ units.
Example: Graph $g(x) = |x + 2|$.

Example:
The absolute value bars indicate that the parent function here is $f(x) = |x|$. Remember that $x + h$ indicates that the graph is shifted to the LEFT not right.

So the graph of $g(x)$ can be formed by taking the graph of $|x|$ and shifting it to the left 2 units.
Example: Graph $F(x) = (x - 1)^3 - 2$

Example:
More than one transformation has been applied to the parent function $f(x) = x^3$ to create $F(x)$.

First, notice that $x$ has been replaced with $x - 1$. So the graph of the cubic function is shifted to the right by 1 unit. Also, the graph has been shifted 2 units downward.

To graph, begin with the cubic parent function.
Start the transformation with the point $(0,0)$ on the graph of $f(x)$. This point will be shifted to $(1, -2)$.
Continue shifting the remaining points to arrive at the graph of $F(x)$. 
The graph of a function can be reflected about the x- and y-axes.
If \( x \) is replaced with \(-x\), then the graph of \( f(x) \) is reflected about the y-axis. So the x-value of each point in the original function changes sign. If \( f(x) \) is multiplied by -1, then the graph of \( f(x) \) is reflected about the x-axis. The y-value of each point in the original function changes sign.
Example: Graph $g(x) = \sqrt{-x}$.

The graph of $g(x)$ can be found by taking the graph of the parent function $f(x) = \sqrt{x}$ and reflecting the graph about the y-axis.
To reflect the graph of $f(x)$ about the y-axis, the sign of the x-values of each of the points needs to be changed. Notice that $(0,0)$ will not be changed and how the domain has changed from the parent function.
Example: Graph \( F(x) = -|x| + 3 \).

Example:
There are two transformations that need to be considered. The graph of the absolute value function has been reflected about the y-axis and then shifted up by a factor of 3.
To graph, draw a chart first, listing the points of the parent function first and then their corresponding points in the graph of F(x) after the transformation.
The graph of a function can be stretched or compressed vertically. If \( f(x) \) is multiplied by a constant \( c \), where the value of \( c \) is greater than 1, then the graph of \( f(x) \) is stretched vertically by a factor of \( c \). In other words, the \( y \)-value of each point on the original graph is multiplied by \( c \). If \( f(x) \) is multiplied by a constant \( c \), where the value of \( c \) is between 0 and 1, then the graph of \( f(x) \) is compressed vertically by a factor of \( c \).
Example: Graph $V(x) = \frac{1}{2} \sqrt[3]{x}$.

Example:
Here the parent function that has been transformed is the cube root function.
Since the constant multiplier is between 0 and 1, this is a vertical compression.
To complete the sketch of $V(x)$, multiply the y-value of each of the points in the cube root function by $\frac{1}{2}$. 
**Example:** Graph \( K(x) = -3x^2 \).

Example:
The parent function here is the square function. The factor of -3 actually represents two transformations: a stretch vertically by a factor of three, and a reflection about the x-axis. Since the graph is being stretched vertically by a factor of three and then reflected about the x-axis, the y-value of each point in the parent function can be multiplied by -3 to get the coordinates of the points in \( K(x) \).
Transformations need to be performed in a particular order in order to arrive at the correct graph for the transformed function. Remember that if it is difficult to complete the transformation directly on the graph, use a table to transform the individual points in the graph. Or draw a new graph corresponding to each step of the transformation so that the process is not quite so overwhelming.

Order of Transformations

1. Reflection about y-axis
2. Horizontal shift
3. Vertical stretch or compression
4. Reflection about x-axis
5. Vertical shift
Example: Graph \( F(x) = 2\sqrt{-x} - 1 \).

Example:

Begin as always by looking at the parent function: \( f(x) = \sqrt{x} \). The three transformations are: reflection about y-axis, stretch vertically by a factor of 2 and shift up 1 unit.

So to complete these transformations and obtain the graph of \( F(x) \), do the following to each of the points in the parent function: change the sign of the \( x \)-value, multiply the \( y \)-value by 2 and subtract 1 from the \( y \)-value.

Now using these points, create the graph of \( F(x) \).
Example: Graph $K(x) = -|x - 2| + 3$.

Example:
The parent function here is $f(x) = |x|$. Again we have a total of three transformations: shift right 2 units, reflection about the x-axis and shift up 3 units.
Example: Construct the equation of the function $g(x)$ that is formed when the following transformations are applied to the function $f(x) = \sqrt[3]{x}$.

- Shifted right 7 units.
- Compressed vertically by a factor of $\frac{2}{3}$.
- Shifted up 4 units.

Example:
The equation of a function can be completed after transformations have been applied.